On the Inconsistency of Classical Logic

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Abstract: This is well-known fact that the classical propositional calculus (zero-order logic, classical propositional logic), is the most fundamental two-valued logical system. This is required for construction of the classical calculus of quantifiers (classical calculus of predicates, first-order logic), which is necessary to construct the classical functional calculus. This last one is needed to formalize the Arithmetic System. At the beginning, we introduce a notation and we repeat some well-known notions (among others, the notions of: operation of consequence, a system, consistency in the traditional sense, consistency in the absolute sense). Next, we present the theorem saying that classical propositional calculus is an inconsistent theory.

Key words: classical propositional calculus, consistency in the traditional sense, consistency in the absolute sense

1. Introduction

The symbols: →, ~, ∨, ∧, ≡ denote the connectives of implication, negation, disjunction, conjunction and equivalence, respectively. \( N = \{1, 2, \ldots\} \) denotes the set of all natural numbers.

Next, \( A_0 = \{p_0^1, p_0^2, \ldots, p_0^k, \ldots\} \) denotes the set of all propositional variables. The symbol \( S_0 \) denotes the set of all well-formed formulas, which are built in the usual manner from propositional variables by means of logical connectives. Next, \( P_0(\phi) \) denotes the set of all propositional variables occurring in \( \phi (\phi \in S_0) \).

\( R_{S_0} \) denotes the set of all rules over \( S_0 \). \( E(\mathcal{M}) \) is the set of all formulas valid in the matrix \( \mathcal{M} \). The symbol \( \mathcal{M}_2 \) denotes the classical two-valued matrix and \( Z_2 \) is the set of all formulas valid in the matrix \( \mathcal{M}_2 \) (see [10], cf. [1 - 7], [11 - 13]). The symbols \( \Rightarrow, \rightarrow, \forall, \exists \) are metalogical symbols.

Next, \( S_0^0 = \{\phi \in S_0 : \phi \notin Z_2 \ & \ & \& \phi \notin Z_2\} \).

Next, \( r_0 \) is the symbol of Modus Ponens in propositional calculus. Hence, \( R_0 = \{r_0\} \). The formula \( X \subseteq Y \) denotes that \( X \subseteq Y \) and \( X \neq Y \). For any \( X \subseteq S_0 \) and \( R \subseteq R_{S_0} \), \( Cn(R, X) \) is the smallest subset of \( S_0 \), containing \( X \), and closed under the rules belonging to \( R \), where \( R \subseteq R_{S_0} \).

The couple \( (R, X) \) is called as a system, whenever \( R \subseteq R_{S_0} \), and \( X \subseteq S_0 \). Hence, \( (R_0, Z_2) \) denotes the system of the classical propositional calculus.

Now we repeat some well-known definitions (see [10], cf. [5, 7 – 9, 11]). Let \( R \subseteq R_{S_0} \) and \( X \subseteq S_0 \). Then:

**Definition 1.1.** \( (R, X) \in Cns^T \iff (\neg \exists \alpha \in S_0) [\alpha \in Cn(R, X) \ & \ & \& \neg \alpha \in Cn(R, X)] \).

**Definition 1.2.** \( (R, X) \in Cns^A \iff Cn(R, X) \neq S_0 \).

\( (R, X) \in Cns^T \) denotes that the system \( (R, X) \) is consistent in the traditional sense. \( (R, X) \in Cns^A \) denotes that the system \( (R, X) \) is consistent in the absolute sense (see [10], cf. [11]).

2. The Main Result

**Theorem.** \( (R_0, Z_2) \notin Cns^T \). (see [15], cf. [14]).

**Proof.** Elementary. □
References


